

Exact and Explicit Optimal Solutions for Trajectory Planning and Control of Single-Link Flexible-Joint Manipulators*

Guanrong Chen†

Department of Electrical and Computer Engineering

Rice University

Houston, Texas 77251

Abstract. *In this paper, an optimal trajectory planning problem for a single-link flexible-joint manipulator is studied. A global feedback-linearization technique is first applied to formulate the nonlinear inequality-constrained optimization problem in a suitable way. Then, an exact and explicit structural formula for the optimal solution of the problem is derived and the solution is shown to be unique. It turns out that the optimal trajectory planning and control can be done off-line, so that the proposed method is applicable to both theoretical analysis and real-time tele-robotics control engineering.*

1. Introduction

An optimal inequality-constrained trajectory planning problem for a standard single-link flexible-joint manipulator is studied in this paper.

From a structural point of view, a robot arm is a weakly-coupled multi-link mechanical transmission chain. Hence, the study of a single-link manipulator (a unit of a robot arm or an independent mechanism) is of fundamental importance.

It is well known that a trajectory planning problem for a flexible-joint manipulator has a nonlinear model. If we consider such a trajectory planning problem under certain additional optimality criterion, then we will encounter a constrained nonlinear optimization problem. No analytic closed-form optimal solution can be found for such problems in general. However, for a single-link flexible-joint manipulator with single control input, Marino and Spong (1986) shown that a nonlinear feedback configuration can be designed to linearize the non-

* Supported by ONR Contract N00014-88-K-0127.

† The author will join the Department of Electrical Engineering, University of Houston in September, 1990.

linear system globally. The basic idea is, roughly speaking, that one can find a nonlinear feedback to "cancel" the nonlinearity of the system and obtain a linear plant and a linear feedback, leaving the nonlinearity to an explicit transformation. The advantage of this approach is that the final result is exact (no linearization error) after a nonlinear inverse transform. This mathematical technique is of course well known in nonlinear control theory (see, for example, Isidori (1989)). Nevertheless, based on this result, we show in this paper that if we consider a minimum control-energy criterion for the trajectory planning (with inequality-constraints) of such manipulators, then an explicit formulation of the optimal solution for the overall nonlinear constrained optimization problem can be obtained in closed form.

The proposed new approach for obtaining an exact optimal solution explicitly for such an inequality-constrained nonlinear optimization problem is novel in mathematics and very useful in robotics engineering since it provides us an analytic solution before the control process is started, so that no on-line computer is needed in the real-time applications (unless the environment is changing and needs to be adapted), which is sometimes impossible in certain control processing such as in some tele-robotics control in aerospace engineering. Another advantage of closed-form solutions over numerical solutions is the convenience in theoretical analysis of the optimal trajectory planning. Even if in the case that the resultant analytic optimal trajectory cannot be traced by actual control inputs, we know the exact optimal trajectory to be approximated.

This paper is organized as follows: We first describe the optimal trajectory planning problem for a single-link flexible-joint manipulator. Then, we use a standard global feedback-linearization technique to formulate the nonlinear inequality-constrained optimization problem in a suitable way. Based on this mathematical model, we finally give an explicit structural solution for the problem in a closed-form.

2. Description of the Problem

Consider a single-link flexible-joint manipulator as shown in Figure 1.

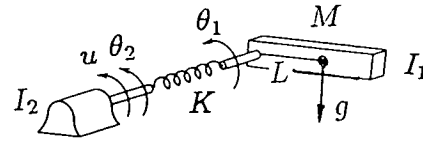


Figure 1. A single-link flexible-joint manipulator

Damping will be ignored in this system for simplicity. The joint is assumed to be of revolute type and the link is assumed to be rigid with inertia I_1 about the axis of rotation. Let θ_1 be the link-angular variable and θ_2 the actuator-shaft angle. Suppose that the rotor inertia of the actuator is I_2 . Assume also that the flexible joint is modeled as a linear spring of stiffness K . Then, by the Euler-Lagrange equations we have the following motion equations for this manipulator:

$$\begin{cases} I_1 \ddot{\theta}_1 + M g L \sin(\theta_1) + K(\theta_1 - \theta_2) = 0 \\ I_2 \ddot{\theta}_2 - K(\theta_1 - \theta_2) = u, \end{cases} \quad (1)$$

where M is the total mass of the link, L the distance from the mass-center of the link to the axis of the rotation, g the acceleration constant of gravity, and u the (generalized) force-input applied to the shaft by the actuator.

Since from a mathematical point of view there is no difference between bend and swivel joints (see, for example, Section 5.3 in Nagy and Siegler (1987)), the problem under investigation has rather wide applications.

We will consider an optimal point-to-point trajectory planning problem for this model. To describe the problem more precisely, let $p = p(t)$, $v = v(t)$, $a = a(t)$, $j = j(t)$ be the position, velocity, acceleration, and jerk of the link, respectively, which are functions of the time variable $t \in [0, T]$ for some fixed terminal time $T < \infty$. The first objective is to design a control input $u = u(t)$ to drive the link such that

$$\begin{aligned} p_i &\leq p(t_i) \leq \bar{p}_i, & v_i &\leq v(t_i) \leq \bar{v}_i, \\ a_i &\leq a(t_i) \leq \bar{a}_i, & j_i &\leq j(t_i) \leq \bar{j}_i, \end{aligned} \quad (2)$$

$i = 0, 1, \dots, n$, for some pre-assigned constants

$$p_i, \bar{p}_i, v_i, \bar{v}_i, a_i, \bar{a}_i, j_i, \bar{j}_i: \quad i = 0, 1, \dots, n,$$

at the pre-desired time instants

$$0 \leq t_0 < t_1 < \dots < t_n \leq T.$$

It can be easily seen from the above trajectory constraints that we will have infinitely many solutions that satisfy the requirements. We want to find an optimal one from them. For this purpose, we consider the problem of controlling the link to satisfy the above trajectory constraints while minimizing certain control-energy to be described precisely later in the next section.

This consideration is especially important when the link is heavy with a large mass M and the control energy (power source) is limited, such as in some aerospace engineering applications.

A direct approach for formulating and solving such a nonlinear inequality-constrained optimization problem does not seem to be easy, unless numerically. However, as mentioned above, numerical solutions are undesirable if analytic solutions can be easily obtained, in particular, for the purpose of analysis of the control system. In the following two sections, we will first formulate the problem in a suitable way and then derive an closed-form structure for the optimal solution. The resultant optimal solution is actually exact in the sense that no approximation will have been applied.

3. Mathematical Formulation of the Problem

In order to formulate the above-described nonlinear inequality-constrained optimization problem in a suitable way, we first rewrite the motion equations in a state-vector setting and then verify that the resultant nonlinear system satisfies some necessary and sufficient condi-

tions so that it can be linearized globally by a feedback, in the sense that an equivalent but linear closed-loop results. All analyses given in this section are standard in nonlinear systems control theory (see, again, Isidori (1989)) and, in fact, have been done in Marino and Spong (1986) (see, also, Spong and Vidyasagar (1989)) for this particular manipulator model. This technique was also used by Tarn et al (1987).

Let

$$x_1 = \theta_1, \quad x_2 = \dot{\theta}_1, \quad x_3 = \theta_2, \quad x_4 = \dot{\theta}_2,$$

so that equations (1) can be rewritten as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u, \quad (3)$$

where $\mathbf{x} = [x_1 \ x_2 \ x_3 \ x_4]^T$, $\mathbf{g}(\mathbf{x}) = [0 \ 0 \ 0 \ I_2^{-1}]^T$, and

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} x_2 \\ -I_1^{-1}MgL \sin(x_1) - I^{-1}K(x_1 - x_3) \\ x_4 \\ I_2^{-1}K(x_1 - x_3) \end{bmatrix}.$$

For this nonlinear system, the vector fields $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ are smooth, the corresponding Lie brackets $[\mathbf{f}, \mathbf{g}] := \frac{\partial \mathbf{g}}{\partial \mathbf{x}} \mathbf{f} - \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \mathbf{g}$ are given by

$$\begin{aligned} &\{ \mathbf{g}, [\mathbf{f}, \mathbf{g}], [\mathbf{f}, [\mathbf{f}, \mathbf{g}]], [\mathbf{f}[\mathbf{f}, [\mathbf{f}, \mathbf{g}]]] \} \\ &= \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ I_2^{-1} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ I_2^{-1} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ I_1^{-1}I_2^{-1}K \\ 0 \\ -I_2^{-2}K \end{bmatrix}, \begin{bmatrix} I_1^{-1}I_2^{-1}K \\ 0 \\ -I_2^{-2}K \\ 0 \end{bmatrix} \right\}, \end{aligned}$$

the vector fields

$$\{ \mathbf{g}, [\mathbf{f}, \mathbf{g}], [\mathbf{f}, [\mathbf{f}, \mathbf{g}]] \}$$

are constant and hence form an involutive set, and moreover the vector fields

$$\{ \mathbf{g}, [\mathbf{f}, \mathbf{g}], [\mathbf{f}, [\mathbf{f}, \mathbf{g}]], [\mathbf{f}[\mathbf{f}, [\mathbf{f}, \mathbf{g}]]] \}$$

are linearly independent for all $0 < K, I_1, I_2 < \infty$. Hence, it follows from a result of Su (1981) that the nonlinear system (3) is globally feedback-linearizable, in the sense

that an equivalent linear feedback system with an explicit nonlinear inverse transform exists. More precisely, we have the following analysis:

Let $\nabla(\cdot) : R \rightarrow R^4$ be the gradient vector of the scalar-valued argument and $\langle \cdot, \cdot \rangle$ a standard inner-product of vector-valued functions. Set $y = [y_1 \ y_2 \ y_3 \ y_4]^T$ with

$$\begin{cases} y_1 = x_1 \\ y_2 = \langle \nabla(y_1), f \rangle = x_2 \\ y_3 = \langle \nabla(y_2), f \rangle = -I_1^{-1} M g L \sin(x_1) \\ \quad - I_1^{-1} K(x_1 - x_3) \\ y_4 = \langle \nabla(y_3), f \rangle = -I_1^{-1} M g L x_2 \cos(x_1) \\ \quad - I_1^{-1} K(x_2 - x_4). \end{cases} \quad (4)$$

Then, with the linearizing feedback control of the form

$$u = F(x) + I_1 I_2 K^{-1} v, \quad (5)$$

where

$$F(x) = I_1^{-1} M g L \sin(x_1) [x_2^2 + I_1^{-1} M g L \cos(x_1) + I_1^{-1} K] \\ + I_1^{-1} K(x_1 - x_3) [(I_1^{-1} + I_2^{-1}) K + I_1^{-1} M g L \cos(x_1)],$$

the nonlinear system (3) has been linearized as

$$\dot{y} = Ay + bv, \quad (6)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

with the following physical meanings:

$$\begin{aligned} y_1 = x_1 &= \text{position of the link} \\ y_2 = x_2 &= \text{velocity of the link} \\ y_3 = \dot{y}_2 &= \text{acceleration of the link} \\ y_4 = \dot{y}_3 &= \text{jerk of the link.} \end{aligned}$$

The original nonlinear control system and the equivalent closed-loop and linearized feedback configuration of the overall system are shown and compared in Figure 2 below.

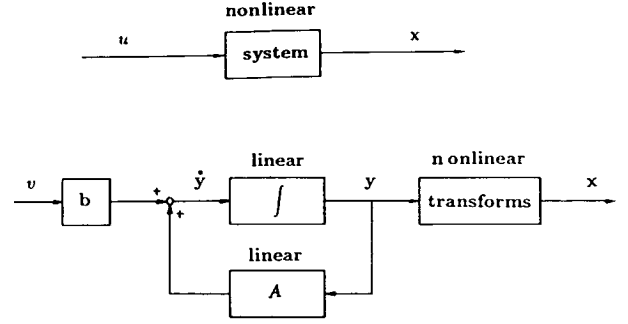


Figure 2. Equivalent Feedback Loops

Here, it is important to point out that if we only consider the trajectory planning (constraints (2)), then the systems shown in Figure 2 are equivalent in the sense that if the trajectory of the linearized feedback system can be controlled to satisfy the constraints (2), then the same can be done for the original nonlinear system by inverting the nonlinear transform (4). For this reason, from now on we can leave the original nonlinear system and work on the linear system (6) together with the nonlinear feedback (5) instead.

Note that in the linearized feedback system, v is the only external and active control input, as can be seen from Figure 2 above. Hence, in the study of the trajectory planning for the linearized feedback system instead of the original nonlinear system, we may consider to minimize the total control-energy of this executive input v . Based on this point of view, we formulate an optimal trajectory planning problem as follows:

Problem:

$$\min_{v \in L_2(0, T)} \int_0^T v^2(t) dt \quad (7a)$$

subject to the linear system

$$\dot{y} = Ay + bv \quad (7b)$$

and the trajectory constraints

$$\begin{aligned} \underline{p}_i &\leq p(t_i) \leq \bar{p}_i, & \underline{v}_i &\leq v(t_i) \leq \bar{v}_i, \\ \underline{a}_i &\leq a(t_i) \leq \bar{a}_i, & \underline{j}_i &\leq j(t_i) \leq \bar{j}_i, \end{aligned} \quad (7c)$$

$$i = 0, 1, \dots, n, \text{ where } 0 \leq t_0 < t_1 < \dots < t_n \leq T < \infty.$$

Here, $L_2(0, T)$ denotes the standard Hilbert space of square-integrable real-valued functions defined on the time-interval $[0, T]$. Once we have solved the linear constrained optimization problem (7a-c) for optimal v^* and $\mathbf{y}^* = [y_1^* \ y_2^* \ y_3^* \ y_4^*]^\top$, we obtain an optimal solution for the original problem from the inversion of (4); namely, from the following formulas:

$$\begin{cases} \mathbf{x}^* = \begin{bmatrix} y_1^* \\ y_2^* \\ y_1^* + I_1 K^{-1} [y_3^* + I_1^{-1} M g L \sin(y_1^*)] \\ y_2^* + I_1 K^{-1} [y_4^* + I_1^{-1} M g L y_2^* \cos(y_1^*)] \end{bmatrix} \\ u^* = F(\mathbf{x}^*) + I_1 I_2 K^{-1} v^* \end{cases} \quad (8)$$

In the next section, we will show a closed-form structure for the optimal solution of the inequality-constrained optimization problem (7a-c).

4. Closed-Form Optimal Solution

In this section, we show the closed-form structure of the optimal solution for the inequality-constrained optimal trajectory planning problem formulated in (7a-c) above. As a result, the optimal solution for the original nonlinear inequality-constrained optimization problem turns out to be exact in an explicit closed-form.

In order to state our result precisely, we need some new notations. In addition to the notations used before, set the matrix-valued exponential function

$$e^{(t-\tau)} \begin{bmatrix} 0 & I \\ A^\top A & A - A^\top \end{bmatrix} := \begin{bmatrix} E_1(t-\tau) & E_2(t-\tau) \\ E_3(t-\tau) & E_4(t-\tau) \end{bmatrix},$$

$0 \leq \tau, t \leq T$, in which each submatrix $E_i(t-\tau)$, $i = 1, 2, 3, 4$, is a 4×4 block. Then, using the notation $\mathbf{1} := [1 \ 1 \ 1 \ 1]^\top$, define

$$\mathbf{h}(t-\tau) = E_2(t-\tau)\mathbf{1}$$

and

$$\mathbf{h}_+(t-\tau) = \begin{cases} E_2(t-\tau)\mathbf{1} & t \geq \tau \\ 0 & t < \tau \end{cases}$$

Moreover, let

$$\dot{\mathbf{h}}(t-\tau) = \frac{\partial}{\partial t} \mathbf{h}(t-\tau)$$

and

$$\dot{\mathbf{h}}_+(t-\tau) = \begin{cases} \frac{\partial}{\partial t} \mathbf{h}(t-\tau) & t \geq \tau \\ 0 & t < \tau \end{cases}$$

Our main result can now be stated as follows:

Theorem. *The optimal solution for Problem (7a-c) is given by*

$$\begin{cases} \mathbf{y}^*(t) = C_{01} \mathbf{h}_+(t-t_0) + C_{02} \dot{\mathbf{h}}_+(t-t_0) \\ \quad + \sum_{i=1}^{n-1} C_i \mathbf{h}_+(t-t_i) \\ v^*(t) = \dot{y}_4^*(t), \end{cases} \quad (9)$$

where $C_{01}, C_{02}, C_i, i = 1, \dots, n-1$, are all 4×4 diagonal constant matrices which are uniquely determined by the trajectory constraints (7c) from the given data set

$$\underline{p}_i, \bar{p}_i, \underline{v}_i, \bar{v}_i, \underline{a}_i, \bar{a}_i, \underline{j}_i, \bar{j}_i, : \quad i = 0, 1, \dots, n. \quad (10)$$

Consequently, the optimal solution (u^*, \mathbf{x}^*) for the original problem is obtained via (8) from the optimal solution (v^*, \mathbf{y}^*) given by (9).

We remark that the determination of the constant coefficient matrices of $\mathbf{y}^*(t)$ is simple, which can be done easily by using any standard quadratic programming algorithm even before the manipulator control processing is started, so that no on-line computer is needed for this optimal nonlinear trajectory planning problem unless adaptive control is necessary. More precisely, we demonstrate this procedure as follows: First, we observe that the minimization problem

$$\min_{v \in L_2(0, T)} \int_0^T v^2(t) dt$$

is equivalent to either

$$\min_{v \in L_2(0, T)} \int_0^T [\mathbf{b}v]^\top [\mathbf{b}v] dt$$

or

$$\min_{\mathbf{y} \in H_1(0, T)} \int_0^T [\dot{\mathbf{y}} - \mathbf{A}\mathbf{y}]^\top [\dot{\mathbf{y}} - \mathbf{A}\mathbf{y}] dt, \quad (11)$$

where $H_1(0, T)$ is the standard first order Sobolev space. If we can solve the minimization problem (11) for \mathbf{y}^* ,

then we can find the optimal solution $v^* = \dot{y}_4^*$ (see (6)). Secondly, we notice that the minimization problem (11) together with the inequality-constraints (10) can be reformulated as the following quadratic programming problem:

$$\min_{C_{01}, C_{02}, \{C_i\}} C^T W C \quad (12)$$

subject to

$$p_i \leq p(t_i) \leq \bar{p}_i, \quad v_i \leq v(t_i) \leq \bar{v}_i,$$

$$a_i \leq a(t_i) \leq \bar{a}_i, \quad j_i \leq j(t_i) \leq \bar{j}_i,$$

$i = 0, 1, \dots, n$, where C is a constant vector consisting of all elements of the diagonal matrices $C_{01}, C_{02}, C_1, \dots, C_n$ and W is a constant matrix consisting of the integrations of all elements of the functions h_+ and \dot{h}_+ . Both C and W have simple explicit expressions as can be easily seen and derived from formulas (9) and (11). This standard quadratic programming problem can be solved by some existing computer routines, which will provide us the unique optimal solution.

5. Conclusions

In this paper, we have studied an optimal trajectory planning problem of a standard single-link flexible-joint manipulator. We first used a standard feedback-linearization technique to formulate the nonlinear inequality-constrained optimization as a minimum control-energy problem. Then, we have derived an exact structural for-

mula in closed-form for the optimal solution of the problem and showed that this solution is unique. The proposed approach is applicable to both theoretical analysis and real-time robotics control engineering.

REFERENCES

- [1] A. Isidori (1989): *Nonlinear Control Systems: An Introduction*, second edition, Springer-Verlag, New York.
- [2] R.W. Marino and M.W. Spong (1986): Nonlinear control techniques for flexible joint manipulators: A single link case study, Proc. 1986 IEEE Conf. Robotics and Automation, San Francisco, April 1986, pp.1030-1036.
- [3] F.N. Nagy and A. Siegler (1987): *Engineering Foundations of Robotics*, Prentice-Hall, Englewood Cliffs, New Jersey.
- [4] M.W. Spong and M. Vidyasagar (1989): *Robot Dynamics and Control*, Wiley, New York.
- [5] R. Su (1981): On the linear equivalents of nonlinear systems, Sys. Contr. Letts., Vol.2, pp.48-52.
- [6] T.J. Tarn, A.K. Bejczy, and X. Yun (1987): Nonlinear feedback control of multiple robot arms, Proc. NASA Workshop on Space Telerobotics, Jan. 20-22, 1987, Pasadena, CA.